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Symmetries of the S matrix for massless particles

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Abstract. Within the framework of local relativistic quantum field theory we consider generators Q of symmetry transformations acting additively on asymptotic particle states according to a given equation. If there is a mass gap this equation can be derived for Q defined as an integral over a conserved local current. For simplicity, we consider only the case that all asymptotic fields are scalar ones. Assuming that the generator Q in this equation is also well defined for massless particles and that elastic scattering occurs at least in an open subset of the scattering manifold, it is shown that Q is at most a linear combination of the generators of the conformal group and internal symmetries. Using the result that massless particles do not interact in a local, dilatationally invariant quantum field theory the conformal group in our result has to be restricted to the Poincaré group. Analogously to the massive case, Q is therefore at most a linear combination of the generators of the Poincaré group and internal symmetries.

1. Introduction

As shown by Garber and Reeh (1979a) a generator Q representing an additive conservation law in an asymptotically complete local relativistic quantum field theory without zero mass particles and with a mass gap above the vacuum is characterised by

$$\mathrm{i}[Q, \psi_\kappa^{\mathrm{ex}}(x)] = P_{\kappa\lambda}(x, \partial_x)\psi_\lambda^{\mathrm{ex}}(x), \qquad Q\Omega = 0 \tag{1.1}$$

(summation with respect to λ). Here, $\psi_\kappa^{\mathrm{ex}}(x)$ ('ex' stands for 'in' or 'out') are free asymptotic fields of mass m_κ, Ω is the vacuum vector and $P_{\kappa\lambda}$ are polynomials in $x \in \mathbb{R}^4$ and derivatives $\partial_x = (\partial/\partial x^\nu)$, vanishing for unequal masses $m_\lambda \neq m_\kappa$. That is Q on one-particle states commutes with the mass operator. The $P_{\kappa\lambda}$ do not depend on the index ex; therefore Q commutes with the S matrix.

Garber and Reeh (1979b) investigated the case of field theories having only scalar asymptotic fields and, starting from (1.1) and under the assumption that all asymptotic particles have elastic interaction with each other, it was shown that Q can only be a linear combination of the ten generators of the Poincaré group and the generators of internal symmetries. In the present paper, the case is considered that the scalar asymptotic fields $\phi^{\mathrm{ex}}(x)$ have zero mass. Concerning the interaction, we assume for the zero mass particles:

(i) For each particle there is another particle such that elastic scattering occurs between them on some open subset V of the set of momenta allowed by energy and momentum conservation.

(ii) The set of zero mass particles does not decompose into subsets which do not interact with each other.

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Further, we assume that Q in (1.1) is also a well defined operator for scalar asymptotic fields of zero mass

$$\mathrm{i}[Q, \phi_k^{\mathrm{ex}}(x)] = P_{kl}(x, \partial_x)\phi_l^{\mathrm{ex}}(x). \tag{1.2}$$

It is our aim to show that P_{kl} is at most a linear combination of the 15 generators of the conformal group and internal symmetries. The proof will be reduced to the special cases of translationally invariant generators and generators for which the associated polynomial P_{kl} is linear in the derivatives.

2. Translationally invariant generators

In this case $P_{kl}(x, \partial_x) =: R_{kl}(\partial_x)$ does not explicitly depend on x

$$\mathrm{i}[Q, \tilde{\phi}_k^{\mathrm{ex}}(p)] = R_{kl}(\mathrm{i}p)\tilde{\phi}_l^{\mathrm{ex}}(p). \tag{2.1}$$

Here, $\tilde{\phi}^{\mathrm{ex}}(p)$ denotes the Fourier transformed fields.

Now the proof of theorem 2.3 in Garber and Reeh (1979b) can be adopted with an obvious change of notation (the solution of the functional equation (2.4) has to be modified slightly) to obtain for a translation invariant generator a result, which is analogous to the massive case.

Theorem 2.1. Let Q be a translation invariant generator

$$\mathrm{i}[Q, \tilde{\phi}_k^{\mathrm{ex}}(p)] = R_{kl}(\mathrm{i}p)\tilde{\phi}_l^{\mathrm{ex}}(p)$$

with scalar asymptotic fields $\tilde{\phi}_l^{\mathrm{ex}}(p)$ of zero mass. Assume that the interaction assumptions (i) and (ii) are fulfilled. Then

$$R_{kl}(\mathrm{i}p) = \delta_{kl}(\boldsymbol{a}\boldsymbol{p} + a^0\omega(\boldsymbol{p})) + c_{kl}$$

with constants $\boldsymbol{a}$, a^0, c_{kl} and $\omega(\boldsymbol{p}) := |\boldsymbol{p}|$.

3. Q as a linear combination of the generators of the conformal group

In the following we consider a generator Q with

$$\mathrm{i}[Q, \phi_k^{\mathrm{ex}}(x)] = \{a_{kl}(x) + b_{kl}^{\mu}(x)\partial_\mu\}\phi_l^{\mathrm{ex}}(x). \tag{3.1}$$

Here, $a_{kl}(x)$ and $b_{kl}(x)$ are polynomials in $x \in \mathbb{R}^4$ so that $G_{kl}(x, \partial_x) := a_{kl}(x) + b^{\mu}{}_{kl}(x)\partial_\mu$ is linear in the derivatives. The property that Q on one-particle states commutes with the mass operator $P_\mu P^\mu$, $[Q, P_\mu P^\mu]\phi_k^{\mathrm{ex}}(x)\Omega = 0$, is equivalent to

$$\partial_\mu\partial^\mu(P_{kl}(x, \partial_x)\phi_l^{\mathrm{ex}}(x)) - (P_{kl}(x, \partial_x)\partial_\mu\partial^\mu\phi_l^{\mathrm{ex}}(x)) = 0 \tag{3.2}$$

with $P_{kl}(x, \partial_x)$ from (1.2).

Commuting (3.2) with $\phi_r^{\mathrm{ex}}(y)$ and using the Schwartz normal form (Schwartz 1961) $[\phi_l^{\mathrm{ex}}(x), \phi_r^{\mathrm{ex}}(y)] = -\mathrm{i}\delta_{lr}\Delta_{(0)}(x-y)$, one obtains

$$0 = [\partial_\mu\partial^\mu, P_{kl}(x, \partial_x)]f(x) \tag{3.3}$$

with every solution f of the Klein-Gordon equation of zero mass. For the special polynomial $G_{kl}(x, \partial_x)$, (3.3) reads

$$0=[\partial_\mu\partial^\mu,\ G_{kl}(x,\partial_x)]f(x). \tag{3.4}$$

Lemma 3.1. Any polynomial $G_{kl}(x, \partial_x)$ obeying (3.4) can be written as a linear combination of ∂_ρ, $x_\mu\partial_\nu - x_\nu\partial_\mu$, $x^\alpha\partial_\alpha$, $2x_\beta x^\nu\partial_\nu + 2x_\beta - x^2\partial_\beta$ and an arbitrary constant.

Proof. For fixed but arbitrary indices k, l (3.4) reads

$$\begin{aligned}0=[\partial_\mu\partial^\mu,\ G(x,\partial_x)]f(x) &= \{(\partial_\mu\partial^\mu G(x,\partial_x))+2(\partial^\mu G(x,\partial_x))\partial_\mu\}f(x)\\ &=\{(\partial_\mu\partial^\mu a(x))+[2(\partial^\mu a(x))+(\partial_\alpha\partial^\alpha b^\mu(x))]\partial_\mu\\ &\quad+(\partial^\alpha b^\beta(x)+\partial^\beta b^\alpha(x))\partial_\alpha\partial_\beta\}f(x).\end{aligned}$$

Since an arbitrary multiple of $\partial_\mu\partial^\mu$ is the only annihilator of every solution $f(x)$ of $\partial_\alpha\partial^\alpha f(x)=0$, we obtain, after comparing the coefficients of the differential operator in the curly bracket, a system of linear partial differential equations for the polynomials $a(x)$ and $b^\mu(x)$

$$\begin{aligned}&a(x)=0\\ &\partial_\alpha\partial^\alpha b^\mu(x)+2\partial^\mu a(x)=0\\ &\partial^\alpha b^\beta(x)+\partial^\beta b^\alpha(x)=\lambda(x)g^{\alpha\beta};\\ &\lambda(x)=2\partial^0 b^0(x)=-2\partial^j b^j(x), \qquad j=1,2,3.\end{aligned}$$

They can be solved and the solution is (see equation (A10))

$$\begin{aligned}&a(x)=2cx+r\\ &b^\mu(x)=2cxx^\mu-x^2c^\mu+2\omega^{\nu\mu}x_\nu+dx^\mu+b^\mu\end{aligned}$$

where $cx=c^\nu x_\nu$, $x^2=x^\nu x_\nu$ and r, d, b^μ, c^μ and $\omega^{\mu\nu}=-\omega^{\nu\mu}$ are 16 real constants. Therefore, G_{kl} in (3.4) can be written as

$$\begin{aligned}G_{kl}(x,\partial_x) &= a_{kl}(x)+b_{kl}(x)\partial_\mu\\ &= c^\mu_{kl}(2x_\mu x^\nu\partial_\nu+2x_\mu-x^2\partial_\mu)+d_{kl}(x^\nu\partial_\nu+1)+b^\mu_{kl}\partial_\mu\\ &\quad+\omega^{\nu\mu}_{kl}(x_\nu\partial_\mu-x_\nu\partial_\mu)+c_{kl}\end{aligned} \tag{3.5}$$

with $c_{kl}=r_{kl}-d_{kl}$ so that lemma 3.1 is proven.

If the coefficients $c^\mu{}_{kl}$, d_{kl}, b^μ_{kl} and $\omega^{\mu\nu}_{kl}$ do not depend on the indices kl, Q would be on one-particle states a linear combination of the generators of the conformal group and internal symmetries c_{kl} where the former are defined for scalar fields ϕ^{ex}_k as follows (see Mack and Salam 1969)

$$\left.\begin{aligned}&\mathrm{i}M_{\mu\nu}\phi^{\mathrm{ex}}_k(x)\Omega=(x_\mu\partial_\nu-x_\nu\partial_\mu)\delta_{kl}\phi^{\mathrm{ex}}_l(x)\Omega\\ &\mathrm{i}P_\rho\phi^{\mathrm{ex}}_k(x)\Omega=\partial_\rho\delta_{kl}\phi^{\mathrm{ex}}_l(x)\Omega\end{aligned}\right\}: \quad \text{10 generators of the Poincaré group}$$

$$\mathrm{i}D\phi^{\mathrm{ex}}_k(x)\Omega=(x^\nu\partial_\nu+1)\delta_{kl}\phi^{\mathrm{ex}}_l(x)\Omega: \quad \text{generator of dilations}$$

$$\mathrm{i}K_\mu\phi^{\mathrm{ex}}_k(x)\Omega=(2x_\mu x^\nu\partial_\nu+2x_\mu-x^2\partial_\mu)\delta_{kl}\phi^{\mathrm{ex}}_l\Omega: \quad \text{four special conformal generators.}$$

4. General case

For a general generator Q in (1.2), assume that $P_{kl}(x, \partial_x)$ has degree N in x. Then an arbitrary generator can be put into the form

$$\mathrm{i}[Q, \phi_k^{\mathrm{ex}}(x)] = \sum_{n=0}^{N} x_{\mu_1} \ldots x_{\mu_n} \partial_{\nu_1} \ldots \partial_{\nu_m} c_{kl}^{\mu_1 \ldots \mu_n ; \nu_1 \ldots \nu_m} \phi_l^{\mathrm{ex}}(x) \tag{4.1}$$

(summation convention!) with coefficients c symmetric in $\mu_1 \ldots \mu_n$ and in $\nu_1 \ldots \nu_m$ respectively.

In order to reduce the degree of P_{kl} in x to $N=2$, we consider at first $Q^{\rho_1 \ldots \rho_N} :=$ $\mathrm{i}[\mathrm{i}[\mathrm{i} \ldots \mathrm{i}[Q, P^{\rho_1}], P^{\rho_2}], \ldots P^{\rho_N}]$, which is translationally invariant

$$\mathrm{i}[Q^{\rho_1 \ldots \rho_N}, \phi_k^{\mathrm{ex}}(x)] = (\partial^{\rho_1} \ldots \partial^{\rho_N} P_{kl}(x, \partial_x)) \phi_l^{\mathrm{ex}}(x) = N! \partial_{\nu_1} \ldots \partial_{\nu_m} c_{kl}^{\rho_1 \ldots \rho_N ; \nu_1 \ldots \nu_m} \phi_l^{\mathrm{ex}}(x). \tag{4.2}$$

Application of theorem 2.1 leads to

$$\mathrm{i}^m p_{\nu_1} \ldots p_{\nu_m} c_{kl}^{\rho_1 \ldots \rho_N ; \nu_1 \ldots \nu_m} = \delta_{kl} \mathrm{i} p_\nu c^{\rho_1 \ldots \rho_N ; \nu} + c_{kl}^{\rho_1 \ldots \rho_N} \qquad \text{on } p^2 = 0. \tag{4.3}$$

Inserting (4.3) into (4.1), we obtain

$$\mathrm{i}[Q, \phi_k^{\mathrm{ex}}(x)] = \Big\{ \delta_{kl} c^{\mu_1 \ldots \mu_N ; \nu} x_{\mu_1} \ldots x_{\mu_N} \partial_\nu + c_{kl}^{\mu_1 \ldots \mu_N} x_{\mu_1 \ldots} x_{\mu_N} + \sum_{n=0}^{N-1} x_{\mu_1} \ldots x_{\mu_n} \partial_{\nu_1} \ldots \partial_{\nu_m} c_{kl}^{\mu_1 \ldots \mu_n ; \nu_1 \ldots \nu_m} \Big\} \phi_l^{\mathrm{ex}}(x). \tag{4.4}$$

Consider now $Q^{\rho_1 \ldots \rho_{N-1}}$ for which

$$\begin{aligned}
\mathrm{i}[Q^{\rho_1 \ldots \rho_{N-1}}, \phi_k^{\mathrm{ex}}(x)] &= \{ N!\, \delta_{kl} c^{\rho_1 \ldots \rho_{N-1} \mu_N ; \nu} x_{\mu_N} \partial_\nu + N!\, c_{kl}^{\rho_1 \ldots \rho_{N-1} \mu_N} x_{\mu_N} \\
&\quad + (N-1)!\, \partial_{\nu_1} \ldots \partial_{\nu_m} c_{kl}^{\rho_1 \ldots \rho_{N-1} ; \nu_1 \ldots \nu_m} \} \phi_l^{\mathrm{ex}}(x) \\
&=: P_{kl}^{\rho_1 \ldots \rho_{N-1}}(x, \partial_x) \phi_l^{\mathrm{ex}}(x).
\end{aligned} \tag{4.5}$$

Inserting $P_{kl}^{\rho_1 \ldots \rho_{N-1}}(x, \partial_x)$ into (3.3) leads to

$$\begin{aligned}
0 &= [\partial_\mu \partial^\mu, \delta_{kl} c^{\rho_1 \ldots \rho_{N-1} \mu_N ; \nu} x_{\mu_N} \partial_\nu + c_{kl}^{\rho_1 \ldots \rho_{N-1} \mu_N} x_{\mu_N}] f(x) \\
&= \{ \delta_{kl} c^{\rho_1 \ldots \rho_{N-1} \mu_N ; \nu} \partial_{\mu_N} \partial_\nu + c_{kl}^{\rho_1 \ldots \rho_{N-1} \mu_N} \partial_{\mu_N} \} f(x),
\end{aligned}$$

and therefore

$$c_{kl}^{\rho_1 \ldots \rho_N} = 0. \tag{4.6}$$

$$c^{\rho_1 \ldots \rho_N ; \nu} + c^{\rho_1 \ldots \rho_{N-1} \nu ; \rho_N} = 2 c^{\rho_1 \ldots \rho_{N-1} 0 ; 0} g^{\rho_N \nu} = -2 c^{\rho_1 \ldots \rho_{N-1} k ; k} g^{\rho_N \nu}, \qquad k = 1, 2, 3. \tag{4.7}$$

By (A9) ($b^{\rho_1 \ldots \rho_{N-1} ; \nu} \equiv c^{\rho_1 \ldots \rho_N ; \nu}$) we get

$$c^{\mu_1 \ldots \mu_N ; \nu} = 0 \qquad \text{for } N > 2. \tag{4.8}$$

Together with (4.6) we obtain from (4.4) for Q in (4.1)

$$\mathrm{i}[Q, \phi_k^{\mathrm{ex}}(x)] = \sum_{n=0}^{N-1} x_{\mu_1} \ldots x_{\mu_n} \partial_{\nu_1} \ldots \partial_{\nu_m} c_{kl}^{\mu_1 \ldots \mu_n ; \nu_1 \ldots \nu_m} \phi_l^{\mathrm{ex}}(x), \qquad N > 2.$$

Repeating the preceding calculation $(N-2)$ times yields a polynomial $P_{kl}(x, \partial_x)$ of degree two in x only

$$\begin{aligned}
\mathrm{i}[Q, \phi_k^{\mathrm{ex}}(x)] &= \{ x_{\mu_1} x_{\mu_2} \partial_{\nu_1} \ldots \partial_{\nu_m} c_{kl}^{\mu_1 \mu_2 ; \nu_1 \ldots \nu_m} + x_{\mu_1} \partial_{\nu_1} \ldots \partial_{\nu_m} c_{kl}^{\mu_1 ; \nu_1 \ldots \nu_m} \\
&\quad + \partial_{\nu_1} \ldots \partial_{\nu_m} c_{kl}^{\nu_1 \ldots \nu_m} \} \phi_l^{\mathrm{ex}}(x).
\end{aligned} \tag{4.9}$$

Using (4.3), (4.6) and

$$c^{\mu_1\mu_2;\nu} = a^{\mu_1}g^{\mu_2\nu} + a^{\mu_2}g^{\mu_1\nu} - a^{\nu}g^{\mu_1\mu_2} \tag{4.10}$$

with $a^0 := c^{000}$, $a^k := c^{k00}$ ((A8) with $b^{\mu_1\mu_2;\nu} \equiv c^{\mu_1\mu_2;\nu}$ and $a^\mu \equiv c^\mu$), (4.5) reads

$$\begin{aligned}\mathrm{i}[Q^\rho, \phi_k^{\mathrm{ex}}(x)] &= \{2!\, \delta_{kl}c^{\rho\mu_2;\nu}x_{\mu_2}\partial_\nu + \partial_{\nu_1}\ldots\partial_{\nu_m}c_{kl}^{\rho;\nu_1\ldots\nu_m}\}\phi_l^{\mathrm{ex}}(x)\\ &= \{2!\, \delta_{kl}[a^\rho x\partial + a_{\mu_2}(x^{\mu_2}\partial^\rho - x^\rho\partial^{\mu_2})] + \partial_{\nu_1}\ldots\partial_{\nu_m}c_{kl}^{\rho;\nu_1\ldots\nu_m}\}\phi_l^{\mathrm{ex}}(x).\end{aligned}$$

Since $M^{\mu_2\rho}$ is a symmetry generator, $\hat{Q}^\rho := Q^\rho - a_{\mu_2}M^{\mu_2\rho}$ is a generator too, with

$$\mathrm{i}[\hat{Q}^\rho, \phi_k^{\mathrm{ex}}(x)] = \{2!\, \delta_{kl}a^\rho x\partial + \partial_{\nu_1}\ldots\partial_{\nu_m}c_{kl}^{\rho;\nu_1\ldots\nu_m}\}\phi_l^{\mathrm{ex}}(x) =: \hat{P}_{kl}^\rho(x, \partial_x)\phi_l^{\mathrm{ex}}(x). \tag{4.11}$$

To show $m \leq 1$ in (4.11) we apply to (4.11) an infinitesimal Lorentz transformation $M_{\alpha\beta}$, which leads to a translationally invariant generator

$$\begin{aligned}\mathrm{i}[\hat{Q}^\rho, M_{\alpha\beta}] &= [\mathrm{i}[\hat{Q}^\rho, M_{\alpha\beta}], \phi_k^{\mathrm{ex}}(x)]\\ &= [(x_\alpha\partial_\beta - x_\beta\partial_\alpha), \hat{P}_{kl}^\rho(x, \partial_x)]\phi_l^{\mathrm{ex}}(x)\\ &= [x_\alpha\partial_\beta - x_\beta\partial_\alpha, \partial_{\nu_1}\ldots\partial_{\nu_m}c_{kl}^{\rho;\nu_1\ldots\nu_m}]\phi_l^{\mathrm{ex}}(x)\\ &= m(g_{\beta\nu_1}\partial_\alpha - g_{\alpha\nu_1}\partial_\beta)\partial_{\nu_2}\ldots\partial_{\nu_m}c_{kl}^{\rho;\nu_1\ldots\nu_m}\phi_l^{\mathrm{ex}}(x).\end{aligned}$$

Theorem 2.1 implies

$$\mathrm{i}^m p_{\nu_1}\ldots p_{\nu_m}c_{kl}^{\rho;\nu_1\ldots\nu_m} = \delta_{kl}\mathrm{i}c^{\rho,\nu}p_\nu + c_{kl}^\rho \qquad \text{on } p^2 = 0. \tag{4.12}$$

Together with (4.10) and $c_{kl}^{\mu_1\mu_2} = 0$ ((4.6) for $N = 2$) this yields for Q in (4.9)

$$\begin{aligned}\mathrm{i}[Q, \phi_k^{\mathrm{ex}}(x)] &= \{\delta_{kl}[2ax(x\partial) - x^2a\partial + c^{\mu_1;\nu}x_{\mu_1}\partial_\nu] + c_{kl}^\mu x_\mu + \partial_{\nu_1}\ldots\partial_{\nu_m}c_{kl}^{\nu_1\ldots\nu_m}\}\phi_l^{\mathrm{ex}}(x)\\ &=: P_{kl}(x, \partial_x)\phi_l^{\mathrm{ex}}(x).\end{aligned} \tag{4.13}$$

Inserting $P_{kl}(x, \partial_x)$ into (3.3), we get

$$(c_{kl}^\mu - \delta_{kl}2a^\mu)\partial_\mu f(x) = 0. \tag{4.14}$$

Now we have to consider two cases:

(i) At least one of the coefficients a^α does not vanish: $a^\rho \neq 0$. Then from (4.14) $c_{kl}^\mu = \delta_{kl}2a^\mu$ and (4.13) reads

$$\mathrm{i}[Q, \phi_k^{\mathrm{ex}}(x)] = \{\delta_{kl}[a^\mu(2x_\mu x\partial + 2x_\mu - x^2\partial_\mu) + c^{\mu;\nu}x_\mu\partial_\nu] + \partial_{\nu_1}\ldots\partial_{\nu_m}c_{kl}^{\nu_1\ldots\nu_m}\}\phi_l^{\mathrm{ex}}(x). \tag{4.15}$$

Using again (3.3), we get

$$[\partial_\mu\partial^\mu, c^{\mu;\nu}x_\mu\partial_\nu]f(x) = 0 \tag{4.16}$$

and by lemma 3.1:

$$c^{\mu;\nu}x_\mu\partial_\nu = dx\partial + \omega^{\mu\nu}(x_\mu\partial_\nu - x_\nu\partial_\mu) \tag{4.17}$$

with $d := c^{00}$ and $\omega^{\alpha\beta} := c^{\alpha;\beta}$, we obtain for Q in (4.15)

$$\begin{aligned}\mathrm{i}[Q, \phi_k^{\mathrm{ex}}(x)] &= \{\delta_{kl}[a^\mu(2x_\mu x\partial + 2x_\mu - x^2\partial_\mu) + \omega^{\mu\nu}(x_\mu\partial_\nu - x_\nu\partial_\mu) + dx\partial]\\ &\qquad + \partial_{\nu_1}\ldots\partial_{\nu_m}c_{kl}^{\nu_1\ldots\nu_m}\}\phi_l^{\mathrm{ex}}(x)\\ &=: P_{kl}(x, \partial_x)\phi_l^{\mathrm{ex}}(x)\end{aligned} \tag{4.18}$$

respectively for $Q^\rho := \mathrm{i}[Q, P^\rho]$:

$$\mathrm{i}[Q^\rho, \phi_k^{\mathrm{ex}}(x)] = \{\delta_{kl}[2a^\rho(x\partial + 1) + 2a_\mu(x^\mu\partial^\rho - x^\rho\partial^\mu) + c^{\rho;\nu}\partial_\nu]\}\phi_l^{\mathrm{ex}}(x).$$

Introducing $a^\rho D := Q^\rho - 2a_\mu M^{\mu\rho} - c^{\rho;\nu}P_\nu$, which is a symmetry generator, we have

$$\mathrm{i}[a^\rho D, \phi_k^{\mathrm{ex}}(x)] = \delta_{kl} 2a^\rho(x\partial + 1)\phi_l^{\mathrm{ex}}(x).$$

Since $a^\rho \neq 0$, it is shown that dilatations are symmetries of the S matrix. To show finally $m \leq 1$ we apply to (4.18) a scale transformation $U(\lambda)\phi_k^{\mathrm{ex}}(x)U^+(\lambda) = \lambda\phi_k^{\mathrm{ex}}(\lambda x)$ to get for the scale transformed generator $Q_\lambda := U(\lambda)QU^+(\lambda)$

$$\begin{aligned}[Q_\lambda, \phi_k^{\mathrm{ex}}(x)] &= P_{kl}(\lambda^{-1}x, \lambda\partial_x)\phi_l^{\mathrm{ex}}(x) \\ &= \{\delta_{kl}[\lambda^{-1}a^\mu(2x_\mu x\partial + 2x_\mu - x^2\partial_\mu) + \omega^{\mu\nu}(x_\mu\partial_\nu - x_\nu\partial_\mu)] \\ &\quad + \lambda^m \partial_{\nu_1} \ldots \partial_{\nu_m} c_{kl}^{\nu_1\ldots\nu_m}\}\phi_l^{\mathrm{ex}}(x).\end{aligned} \tag{4.19}$$

Now consider the generator $Q_\lambda^\infty := \lim_{\lambda\to\infty} \lambda^{-M}Q_\lambda$, where M is the degree of the polynomial $\Sigma_{m=0}^M \lambda^m \partial_{\nu_1} \ldots \partial_{\nu_m} c_{kl}^{\nu_1\ldots\nu_m}$. Q_λ^∞ is a symmetry of the S matrix and translationally invariant

$$\mathrm{i}[Q_\lambda^\infty, \phi_k^{\mathrm{ex}}(x)] = \partial_{\nu_1} \ldots \partial_{\nu_M} c_{kl}^{\nu_1\ldots\nu_M}\phi_l^{\mathrm{ex}}(x).$$

Theorem 2.1 implies $M = 1$ so that $\mathrm{i}^m p_{\nu_1} \ldots p_{\nu_m} c_{kl}^{\nu_1\ldots\nu_m} = \delta_{kl}\mathrm{i}c^\nu p_\nu + c_{kl}$. Therefore, in the first case we obtain from (4.18) the following result

$$\begin{aligned}\mathrm{i}[Q, \phi_k^{\mathrm{ex}}(x)] = \{&\delta_{kl}[a^\mu(2x_\mu x\partial + 2x_\mu - x^2\partial_\mu + d(x\partial + 1) \\ &+ \omega^{\mu\nu}(x_\mu\partial_\nu - x_\nu\partial_\mu) + c^\mu\partial_\mu] + r_{kl}\}\phi_l^{\mathrm{ex}}(x)\end{aligned}$$

where $r_{kl} := c_{kl} - d\delta_{kl}$ are the internal symmetries.

Hence, Q is a linear combination of the generators of the conformal group and internal symmetries. (This result is compatible with that of Haag *et al* (1975), where supersymmetries are considered.) Furthermore, each of the generators of the conformal group is a symmetry of the S matrix. This can be seen from (4.17) after considering $\lim_{\lambda\to 0} \lambda Q_\lambda$, which shows that $a^\mu K_\mu$ is a symmetry of the S matrix and after applying an infinitesimal Lorentz transformation $M_{\alpha\beta}$, which shows that each K_μ is a symmetry of the S matrix.

(ii) In the second case we have to consider $a^\mu \equiv 0$, $\mu = 0, 1, 2, 3$. Then from (4.18)

$$\mathrm{i}[Q, \phi_k^{\mathrm{ex}}(x)] = \{\delta_{kl}[\omega^{\mu\nu}(x_\mu\partial_\nu - x_\nu\partial_\mu) + dx\partial] + \partial_{\nu_1} \ldots \partial_{\nu_m} c_{kl}^{\nu_1\ldots\nu_m}\}\phi^{\mathrm{ex}}(x).$$

Consider $\hat{Q} := Q - \omega^{\mu\nu}M_{\mu\nu}$ for which

$$\mathrm{i}[\hat{Q}, \phi_k^{\mathrm{ex}}(x)] = \{\delta_{kl}dx\partial + \partial_{\nu_1} \ldots \partial_{\nu_m} c_{kl}^{\nu_1\ldots\nu_m}\}\phi_l^{\mathrm{ex}}(x).$$

Analogously to the derivation of (4.12) from (4.11), we obtain $\mathrm{i}^m p_{\nu_1} \ldots p_{\nu_m} c_{kl}^{\nu_1\ldots\nu_m} = \delta_{kl}\mathrm{i}c^\nu p_\nu + c_{kl}$ to get the following result

$$\mathrm{i}[Q, \phi_k^{\mathrm{ex}}(x)] = \{\delta_{kl}[\omega^{\mu\nu}(x_\mu\partial_\nu - x_\nu\partial_\mu) + c^\mu\partial_\mu + d(x\partial + 1)] + r_{kl}\}\phi_l^{\mathrm{ex}}(x)$$

with $r_{kl} := c_{kl} - d\delta_{kl}$.

Therefore, in this case Q is a linear combination of the generators of the Poincaré group, the generator of dilatations and internal symmetries. Furthermore, each of these generators is a symmetry of the S matrix.

In each case we have shown that dilatations are symmetries of the S-matrix. Now, for example, from Buchholz and Fredenhagen (1977) one knows that in a local dilatationally invariant quantum field theory of massless particles the S matrix is trivial. In our calculation where, by assumption, the S matrix is non-trivial, the coefficients a^μ

and d therefore have to vanish so that Q is as in the massive case a linear combination of the generators of the Poincaré group and internal symmetries.

Appendix

Consider the polynomials $a(x)$ and $b^\mu(x)$, $\mu = 0, 1, 2, 3$, fulfilling the following system of linear partial differential equations

$$\partial_\mu \partial^\mu a(x) = 0 \tag{A1}$$

$$\partial_\alpha \partial^\alpha b^\mu(x) + 2\partial^\mu a(x) = 0 \tag{A2}$$

$$\begin{aligned} &\partial^\alpha b^\beta(x) + \partial^\beta b^\alpha(x) = \lambda(x) g^{\alpha\beta}, \\ &\lambda(x) = 2\partial^0 b^0(x) = -2\partial^k b^k(x) \qquad k = 1, 2, 3. \end{aligned} \tag{A3}$$

They can be solved by putting

$$\begin{aligned} a(x) &= a + x_\mu a^\mu + x_{\mu_1} x_{\mu_2} a^{\mu_1\mu_2} + \ldots + x_{\mu_1} \ldots x_{\mu_n} a^{\mu_1\cdots\mu_n} \\ b^\alpha(x) &= b^\alpha + x_\mu b^{\mu,\alpha} + x_{\mu_1} x_{\mu_2} b^{\mu_1\mu_2,\alpha} + \ldots + x_{\mu_1} \ldots x_{\mu_n} b^{\mu_1\cdots\mu_n,\alpha} \end{aligned} \tag{A4}$$

where a, b^α, $a^{\mu_1\cdots\mu_r}$ and $b^{\mu_1\cdots\mu_r,\alpha}$, $r = 1, 2, \ldots, n$ are unknown coefficients, symmetric with respect to permutation of the indices $\mu_1 \ldots \mu_r$ for $r = 2, 3, \ldots, n$, and n is the highest degree of $a(x)$ or $b^\alpha(x)$. Inserting (A4) into (A2) and (A3), we obtain after comparing coefficients

$$b^{\alpha,\beta} + b^{\beta,\alpha} = 2b^{00} g^{\alpha\beta} = -2b^{kk} g^{\alpha\beta} \tag{A5}$$

$$b^{\mu\alpha,\beta} + b^{\mu\beta,\alpha} = 2b^{\mu 0,0} g^{\alpha\beta} = -2b^{\mu k,k} g^{\alpha\beta} \tag{A6}$$

$$b^{\mu_2\cdots\mu_r\alpha,\beta} + b^{\mu_2\cdots\mu_r\beta,\alpha} = 2b^{\mu_2\cdots\mu_r 0,0} g^{\alpha\beta} = -2b^{\mu_2\cdots\mu_r k,k} g^{\alpha\beta} \tag{A7}$$

for $k = 1, 2, 3$ and $r = 3, 4, \ldots, n$.

It can be easily shown that $b^{\mu\alpha,\beta}$ has only four linear independent coefficients $b^{000} =: c^0$ and $b^{k0,0} =: c^k$ and that

$$b^{\mu\alpha,\beta} = c^\mu g^{\alpha\beta} + c^\alpha g^{\mu\beta} - c^\beta g^{\mu\alpha}. \tag{A8}$$

Equation (A7) then implies

$$b^{\mu_2\cdots\mu_r\alpha,\beta} = 0 \qquad \text{for } r = 3, 4, \ldots, n. \tag{A9}$$

Since from (A8) $b^\mu(x)$ is a polynomial of degree two in x, we see from (A2) that $a(x)$ is a polynomial of degree one in x. Then from (A1) and (A2) we get $a^\mu = 2c^\mu$, $a(x) = a + 2cx$. Putting $a =: r$, $b^{00} = -b^{kk} =: d$, $k = 1, 2, 3$ and $b^{\alpha\beta} =: \omega^{\beta\alpha} = -\omega^{\beta\alpha}$, $\alpha \neq \beta$ ((A5)), we get from (A8) the following result

$$\begin{aligned} a(x) &= 2cx + r \\ b^\mu(x) &= 2cxx^\mu - x^2 c^\mu + 2\omega^{\nu\mu} x_\nu + dx^\mu + b^\mu. \end{aligned} \tag{A10}$$

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